Stability of Multidimensional Linear Time-Varying Systems

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This paper focuses attention on the stability of multidimensional linear second-order systems with time-varying coefficients. Five theorems on stability and two theorems on instability of such systems and ten corollaries are derived based on Liapunov's second method. Unlike many stability theorems which involve determination of the characteristic equations and their roots, or transformation to a system of first-order equations (thereby often losing sight of the physical parameters of the system), conditions are imposed herein directly on the physical parameters of the system. The theorems derived generalize many existing theorems on the stability of linear constant-coefficient systems.

Introduction

IT is well-known that the equations describing most dynamical systems are not amenable to closed-form solution. This motivated the qualitative study of differential equations in which the behavior of the solutions is investigated without finding them explicitly. The particular case of linear constant-coefficient systems has been studied extensively and a large amount of literature is now available on such systems. Considerable effort has also been spent on nonlinear systems with constant coefficients.

Hill and Mathieu's work toward the end of the nineteenth century gave rise to a class of problems in which the parameters were varying periodically with time. Floquet formulated a theorem to find stability of linear differential equations with periodic coefficients.

There are a number of systems in which the parameters vary with time arbitrarily. Examples can be found in all disciplines of engineering, physics, economics, ecology, biosystems, demography, etc. One class of problems that motivated this study deals with space stations of the future for which the system parameters (inertia, damping, and stiffness) governing the dynamics undergo continuous change due to fabrication, construction, and deployment of large structures onboard. For the single-degree-of-freedom system with time-varying coefficients, the Sonin-Polya theorem[1] is applicable to study the nature of system response. This was generalized recently to multidimensional systems by Shrivastava.[2] The generalized theorem, which encompasses many existing theorems and puts stability conditions directly in terms of the system parameters and their variations, is simple to apply for systems with symmetric stiffness matrices. For systems with general stiffness matrices, determination of a matrix S which symmetrizes a parameter matrix is a major problem in the application of the theorem. Presently, determination of S is possible only by trial and error.

In this paper, in dealing with a set of linear, second-order ordinary differential equations with time-varying coefficients 1) some theorems and corollaries are derived which are simpler to apply than those in Ref. 2, and 2) some additional theorems are derived which eliminate determination of the S matrix, and conditions are directly imposed on the system parameters and their variations for stability/instability of the system. The derivations are based on Liapunov's second method.[3] Many theorems available in the literature on the stability of linear constant-coefficient systems are particular cases of the five stability theorems, two instability theorems, and ten corollaries presented here.

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System Definition

The system is governed by a general multidimensional second-order linear time-varying vector equation

$$M(t)q'' + D(t)q' + G(t)q' + K(t)q + A(t)q = 0 \tag{1}$$

where the primes denote differentiation with respect to time t, the independent variable; q is the $(n \times 1)$ state vector; M, D, K are $(n \times n)$ real symmetric matrices whose elements are continuous functions of time on $[0,\infty)$ and differentiable; M is positive definite; and G, A are $(n \times n)$ real skew-symmetric matrices whose elements are continuous functions of time on $[0,\infty)$ and differentiable.

For mechanical systems, M generally denotes the inertia matrix, D the damping matrix, G the gyroscopic matrix, K the stiffness matrix, and A represents the circulatory forces.

To simplify the notation, we shall write Eq. (1) as

$$Mq'' + (D+G)q' + (K+A)q = 0 \tag{2}$$

where the explicit time dependence of the elements of these matrices is to be understood unless stated otherwise. Equation (2) can also be written as

$$Mq'' + Bq' + Cq = 0 \tag{3}$$

where B and C are general matrices.

The above notation holds throughout this paper unless otherwise stated. Throughout this paper, the inner product of two vectors is denoted by

$$(x,y) = x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Positive and negative definite matrices are denoted by

$$A > 0 \text{ and } A < 0$$

respectively, whereas

$$A \geq 0 \text{ and } A \leq 0$$

denote positive and negative semidefinite matrices, respectively.

Theorems on the Stability of Linear Time-Varying Systems

Some of the theorems and corollaries on the stability of linear time-varying systems derived during the present investigation are stated and proved in this section. The proofs are based on construction of an appropriate Liapunov function[3]

Theorem 1. Systems governed by

$$Mq'' + (D+G)q' + Kq = 0 \tag{4}$$

are stable if

$$K>0, \qquad M' \leq 0$$

and

$$2MK^{-1}(D+G) - (MK^{-1}M)' \geq 0 \text{ for all } t \in [0,\infty) \tag{5}$$

and asymptotically stable if

$$K>0, \qquad M'<0$$

and

$$2MK^{-1}(D+G) - (MK^{-1}M)' > 0 \text{ for all } t \in [0,\infty) \tag{6}$$

Proof. For the system, define the Liapunov function as

$$V = (q, Mq) + (q', MK^{-1}Mq')$$

By definition, M is positive definite. Therefore, V is positive definite if $MK^{-1}M$ is positive definite. This can be simplified if the following result is used[4]: "If A is an $(n \times n)$ real symmetric matrix, then A is positive-definite if, and only if, B^TAB is positive definite for a nonsingular real $(n \times n)$ matrix B." Or,

$$MK^{-1}M>0 \Leftrightarrow K^{-1}>0 \Leftrightarrow K>0$$

Hence V is positive definite if $M>0$ and $K>0$. Its derivative is

$$V' = 2(q', Mq) + (q, M'q) + 2(q', MK^{-1}Mq'')$$
$$+ (q', [MK^{-1}M]'q')$$

Substituting for Mq'' from Eq. (2) and simplifying,

$$-V' = (q, -M'q)$$
$$+ (q', [2MK^{-1}(D+G) - (MK^{-1}M)']q')$$

$-V'$ is positive semidefinite if the second and third conditions of Eq. (5) are satisfied. Hence, by Liapunov's stability theorem,[3] Eq. (4) is stable.

For asymptotic stability, $(-V')$ must be positive definite. This is true if the second and third conditions of (6) are satisfied. Hence, by Liapunov's asymptotic stability theorem,[3] Eq. (4) is asymptotically stable.

The preceding theorem is derived for a system with a symmetric stiffness matrix. As a corollary to this, the corresponding theorem is derived for a system with a general stiffness matrix.

Corollary 1.1. For systems governed by

$$Mq'' + Bq' + Cq = 0 \tag{7}$$

if there exists a symmetric positive-definite matrix $S(t)$ such that $SM^{-1}C = (SM^{-1}C)^T$, then a set of sufficient conditions for the stability of Eq. (7) is

$$SM^{-1}C>0, \qquad S' \leq 0$$

and

$$2SC^{-1}B - (SC^{-1}M)' \geq 0 \text{ for } t \in [0,\infty) \tag{8}$$

and a set of sufficient conditions for the asymptotic stability of Eq. (7) is

$$SM^{-1}C>0, \qquad S'<0$$

and

$$2SC^{-1}B - (SC^{-1}M)' > 0 \text{ for all } t \in [0,\infty) \tag{9}$$

This is equivalent to the theorems derived in Ref. 2

Proof. The proof follows from writing Eq. (7) as

$$Sq'' + SM^{-1}Bq' + SM^{-1}Cq = 0$$

Corollary 1.2. For the single-degree-of-freedom system

$$q'' + dq' + kq = 0 \tag{10}$$

where d and k are continuous functions of time on $[0,\infty)$ and differentiable, a set of sufficient conditions for the stability of Eq. (10) is

$$k>0$$

$$2dk + k' \geq 0 \text{ for all } t \in [0,\infty) \tag{11}$$

Corollary 1.3. For the systems governed by

$$Mq'' + Bq' + Cq = 0 \tag{12}$$

where M, B, and C are $(n \times n)$ real constant-coefficient matrices, M is symmetric and positive definite, if there exists a symmetric positive-definite, constant-coefficient matrix S_1 such that $S_1M^{-1}C = (S_1M^{-1}C)^T$, then a set of sufficient conditions for stability of Eq. (12) is

$$S_1M^{-1}C>0, \qquad S_1C^{-1}B \geq 0 \tag{13}$$

This agrees with theorem 6.1 of Ref. 5 which, in the present notation, states: A circulatory system in the absence of velocity-dependent forces is stable if, and only if, there exists a symmetric and positive-definite matrix S_1 such that $S_1M^{-1}C$ is symmetric and positive definite.

The above theorem and its corollaries involve the inversion of the stiffness matrix. This can be cumbersome for high-order systems. A theorem that avoids inversion is stated next, which is applicable to a special class of systems in which the inertia matrix is an identity matrix (i.e., $M = I_n$).

Theorem 2. For systems governed by

$$q'' + Bq' + Cq = 0 \tag{14}$$

if there exists a symmetric positive-defintie matrix $S(t)$ such that $SC = (SC)^T$ and C is nonsingular, then a set of sufficient conditions for the stability of Eq. (14) is

$$SC>0, \qquad (C^TSC)' \leq 0$$

and

$$2SCB - (SC)' \geq 0 \text{ for all } t \in [0,\infty) \tag{15}$$

and a set of sufficient conditions for the asymptotic stability of Eq. (14) is

$$SC>0, \qquad (C^TSC)'<0$$

and

$$2SCB - (SC)' > 0 \text{ for all } t \in [0,\infty) \tag{16}$$

Proof. The proof follows from using the Liapunov function

$$V=(q, C^TSCq)+(q', SCq')$$

Next, we state another theorem which avoids both matrix inversion and multiplication for a class of systems.

Theorem 3. Systems governed by Eq. (4) are stable if

$$K>0, \qquad K'\leq 0$$

and

$$2D-M'\geq 0 \text{ for all } t\in[0,\infty] \tag{17}$$

and asymptotically stable if

$$K>0, \qquad K'<0$$

and

$$2D-M'>0 \text{ for all } t\in[0,\infty) \tag{18}$$

Proof. The proof follows from using the Liapunov function

$$V=(q, Kq)+(q', Mq')$$

Corollary 3.1. For systems governed by Eq. (7), if there exists a symmetric positive-definite matrix $S(t)$ such that $SM^{-1}C=(SM^{-1}C)^T$, then a set of sufficient conditions for the stability of Eq. (7) is

$$SM^{-1}C>0, \qquad (SM^{-1}C)'\leq 0$$

and

$$2SM^{-1}B-S'\geq 0 \text{ for all } t\in[0,\infty) \tag{19}$$

and a set of sufficient conditions for the asymptotic stability of Eq. (7) is

$$SM^{-1}C>0, \qquad (SM^{-1}C)'<0$$

and

$$2SM^{-1}B-S'>0 \text{ for all } t\in[0,\infty) \tag{20}$$

Proof. The proof follows from writing Eq. (7) as

$$Sq''+SM^{-1}Bq'+SM^{-1}Cq=0$$

Corollary 3.2. For the single-degree-of-freedom system [Eq. (10)], where d and k are continuous functions of time on $[0,\infty)$ and differentiable, a set of sufficient conditions for the stability of Eq. (10) is

$$k>0, \qquad k'\leq 0$$

and

$$d\geq 0 \text{ for all } t\in[0,\infty) \tag{21}$$

Note: For a single-degree-of-freedom system, according to corollary 1.2, $(2dk+k')\geq 0$, whereas according to corollary 3.2, $k'\leq 0$. If $d=0$, it may appear that the corollaries lead to contradictory conditions, but if we keep in mind that Liapunov's second method provides us with only sufficient conditions, these results are to be interpreted as providing two different regions of stability in the parameter space.

Corollary 3.3. For systems governed by

$$Mq''+(D+G)q'+Kq=0 \tag{22}$$

where M, D, G, K are $(n\times n)$ real constant-coefficient matrices where $M=M^T>0$, $D=D^T$, $G=-G^T$, $K=K^T$, and q is the $(n\times 1)$ state vector, a set of sufficient conditions for the stability of Eq. (22) is

$$K>0, \qquad D\geq 0 \tag{23}$$

This can be seen to be a particular case of the following well-known result[6]: Consider the equation

$$Ax''+2Bx'+Cx=0, \quad x(0)=C^1, \quad x'(0)=C^2$$

If A, B, and C are positive semidefinite and either A or C is positive definite, then

$$\det(\lambda^2A+2\lambda B+C)=0$$

has no roots with positive real parts. If A and C are positive semidefinite and B is positive definite, then the only root with zero real part is $\lambda=0$.

This result also agrees with the first part of the Kelvin-Tait-Chetayev (KTC) theorem[7] which we state in the notation of Ref. 5: An equilibrium position which is stable under purely potential forces (nongyroscopic nondissipative system or the system with $D=G=0$) remains stable with the addition of gyroscopic and dissipative forces. Later in this paper a theorem is derived which matches with the second part of the KTC theorem.

Corollary 3.4. For systems governed by Eq. (12), where M, B, and C are $(n\times n)$ real constant coefficient matrices and $M=M^T>0$, if there exists a symmetric positive-definite, constant-coefficient matrix S_1 such that $S_1M^{-1}C=(S_1M^{-1}C)^T$, then a set of sufficient conditions for the stability of Eq. (12) is

$$S_1M^{-1}C>0, \qquad S_1M^{-1}B\geq 0 \tag{24}$$

In the absence of velocity-dependent forces, this theorem again agrees with theorem 6.1 of Ref. 5 stated earlier.

Stability Theorems Using a New Approach

In the previous section, stability theorems are derived for systems with symmetric or symmetrizable stiffness matrices. Application of the theorems is straight forward for systems with symmetric stiffness matrices, but is more involved for general stiffness matrices consisting of symmetric and skew-symmetric parts. The major difficulty in the application of the theorem is in determining the matrix S. Presently, no procedure is available to generate the S matrix. As suggested by Huseyin[5] for constant-coefficient systems, one may try a diagonal matrix for S first. However, this may work only for a few cases, and normally, one would have to resort to trial and error. Construction of the S matrix therefore requires ingenuity and labor. Some basic questions remain unanswered, such as

1) If the system is stable, must an S matrix always exist?

2) Is the S matrix for a system unique?

In this section, stability theorems are formulated which do not involve determination of the S matrix. It will be seen that the penalty to be paid is investigating a $(2n\times 2n)$ matrix for sign definiteness.

In theorem 3 for the system [Eq. (4)], we used the Liapunov function

$$V=(q, Kq)+(q', Mq')$$

If the same Liapunov function were used for the general system [Eq. (7)], V' would involve the term $-2(q', Aq)$ in addition to the terms existing already. One is now faced with the problem of determining the sign definiteness of the bilinear form

$$U(x,y)=(x,Ax)+2(x,By)+(y,Cy)$$

Towards this end we now state and prove the following lemma.

Lemma. The bilinear form

$$U(x,y) = (x,Ax) + 2(x,By) + (y,Cy)$$

is positive definite if, and only if, the matrix

$$\left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right]$$

is positive definite.

Proof. It may be verified that

$$(x,Ax) + 2(x,By) + (y,Cy) = [x^T | y^T] \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right] \left[\begin{array}{c} x \\ \hline y \end{array}\right]$$

Define a $(2n \times 1)$ vector

$$z = \left[\begin{array}{c} x \\ \hline y \end{array}\right]$$

Then,

$$U(x,y) = z^T \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right] z$$

Hence $U(x,y)$ is positive definite if, and only if

$$\left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right]$$

is positive definite.

At the time of writing this paper, the authors realized that this lemma had been used earlier in a not so explicit form by Mingori[8] for proving stability theorems for constant-coefficient systems with constraint damping. Therefore, the technique used in this section is not really "new."

Theorem 4. Systems governed by Eq. (2) are stable if

$$K > 0$$

$$\left[\begin{array}{c|c} (-M)' & (MK^{-1}A)^T \\ \hline (MK^{-1}A) & 2MK^{-1}(D+G) - (MK^{-1}M)' \end{array}\right] \geq 0 \text{ for all } t \in [0,\infty) \tag{25}$$

and asymptotically stable if

$$K > 0$$

$$\left[\begin{array}{c|c} (-M)' & (MK^{-1}A)^T \\ \hline (MK^{-1}A) & 2MK^{-1}(D+G) - (MK^{-1}M)' \end{array}\right] > 0 \text{ for all } t \in [0,\infty) \tag{26}$$

Proof. Consider the same Liapunov function as in theorem 1, viz.,

$$V = (q,Mq) + (q', MK^{-1}Mq')$$

$$-V' = (q, -M'q) + 2(q, [MK^{-1}A]^T q')$$

$$+ (q', [2MK^{-1}(D+G) - (MK^{-1}M)']q')$$

We get the stated conditions from the expressions for V and V' by applying the lemma.

Theorem 5. Systems governed by Eq. (2) are stable if

$$K > 0$$

$$\left[\begin{array}{c|c} (-K') & (-A) \\ \hline (A) & (2D-M') \end{array}\right] \geq 0 \text{ for all } t \in [0,\infty) \tag{27}$$

and asymptotically stable if

$$K > 0$$

$$\left[\begin{array}{c|c} (-K') & (-A) \\ \hline (A) & (2D-M'') \end{array}\right] > 0 \text{ for all } t \in [0,\infty) \tag{28}$$

Proof. The proof follows from using the Liapunov function

$$V = (q,Kq) + (q', Mq')$$

Instability Theorems

Theorem 6. Systems governed by

$$Mq'' + Dq' + (K+A)q = 0 \tag{29}$$

are unstable if

$$D' - 2K - M'' > 0 \text{ for all } t \in [0,\infty) \tag{30}$$

Proof. Consider the function

$$V = (q, [D-M']q) + 2(q, Mq')$$

Note that V is a sign variable.

$$V' = (q, [D' - 2K - M'']q) + 2(q', Mq')$$

where $M>0$. Hence $V'>0$ if Eq. (30) is satisfied, and using Liapunov's instability theorem,[3] we conclude that Eq. (29) is unstable.

Corollary 6.1. For systems governed by Eq. (3), if there exists a positive-definite symmetric matrix $S_2(t)$ whose elements are continuous functions of time $[0,\infty)$ and are twice differentiable such that $S_2M^{-1}B = (S_2M^{-1}B)^T$, then a sufficient condition for the instability of Eq. (3) is

$$(S_2M^{-1}B)' - 2S_2M^{-1}C - S_2'' > 0 \text{ for all } t \in [0,\infty) \tag{31}$$

Proof. The proof follows writing Eq. (3) as

$$S_2q'' + S_2M^{-1}Bq' + S_2M^{-1}Cq = 0$$

Corollary 6.2 For the single-degree-of-freedom system [Eq. (10)], where d and k are continuous functions of time on $[0,\infty)$ and are differentiable, a sufficient condition for instability of Eq. (10) is

$$d' - 2k > 0 \text{ for all } t \in [0,\infty) \tag{32}$$

Corollary 6.3. If M, B, and C are $(n \times n)$ real constant-coefficient matrices where $M = M^T > 0$, q is the $(n \times 1)$ state vector, if there exists a symmetric positive-definite matrix S_3 such that $S_3M^{-1}B = (S_3M^{-1}B)^T$, then a sufficient condition for the instability of Eq. (12) is

$$S_3M^{-1}C < 0 \tag{33}$$

It may be noted that this result matches with the second part of the KTC theorem. Using the notation of Ref. 5, this states:

An equilibrium position which is unstable under purely potential forces remains unstable with the addition of gyroscopic and dissipative forces if the latter have complete dissipation.

Theorem 7. Systems governed by Eq. (2), where the elements of M are twice differentiable, are unstable if

$$\left[\begin{array}{c|c} D'-2K-M'' & -G \\ \hline G & 2M \end{array}\right] > 0 \text{ for all } t\in[0,\infty) \qquad (34)$$

Proof. We use the same function as in the theorem 6, viz.,

$$V=(q,[D-M']q)+2(q,Mq')$$

$$V'=(q,[D'-2K-M'']q)+2(q,-Gq')+(q',2Mq')$$

and we get Eq. (34) from V and V' by using the lemma.

Example

As an illustration, the simple example of an extending pendulum is taken up.

Consider a simple pendulum of mass m and varying length $l(t)$. Assuming small-angle motion, the equation of motion is

$$\theta''+2(e/l)\theta'+(g/l)\theta=0 \qquad (35)$$

where primes denote differentiation with remaining terms in time, θ is the angle of inclination, l the instantaneous length, $e=dl/dt$, and g is the acceleration due to gravity.

We use corollary 1.2. Here, $d=2(e/l)$ and $k=(g/l)$.

$$k>0$$

or, $g/l>0$ is true provided $l\neq0$.

$$2dk+k'\geq0$$

or, $(3ge/l^2)\geq0$ is true provided $e\geq0$ and $l\neq0$.

It may be concluded that if $e\geq0$ and $l\neq0$ for every $t\in[0,\infty)$, then according to corollary 1.2, Eq. (35) represents a stable motion. This includes the trivial case of $e=0$ (nonextending pendulum) which is known to be stable. A check on the nontrivial case of $e>0$ is more difficult, and we provide only a partial verification as follows.

An approximate solution to this problem for constant extension rate is derived in Ref. 9 using the WKBJ method. The solution of the form

$$\theta=[C_1\cos\psi+C_2\sin\psi]/(l^3g)^{1/4}$$

where C_1 and C_2 are constants and ψ is a function of time.

It can be easily shown that $|C_1\cos\psi+C_2\sin\psi|\leq|C_1|+|C_2|$, i.e., the numerator is bounded above. For a constant extension rate, $l\to\infty$ as $t\to\infty$ and, hence, $\theta\to0$ as $t\to\infty$. The solution is reported to be valid whenever

$$e/(lg)^{1/2}\ll16/(3)^{1/2}$$

Hence, we have a verification in this range for a constant extension rate.

From the physics of the problem, it is clear that when $e<0$ the motion is unstable. Corollary 1.2 suggests that motion could be unstable if $e<0$.

Conclusion

Seven theorems have been proved in this paper. The first of these provides sufficient conditions for the stability of linear time-varying systems with symmetric stiffness matrices. As a corollary to this, conditions for the stability of systems with symmetrizable stiffness matrices are derived, in addition to specializing the results for the single-degree-of-freedom and constant-coefficient cases. For systems with inertia matrix equal to identity, theorem 2 provides stability conditions which do not involve inversion of matrices. Again, for systems with symmetric stiffness matrices, theorem 3 provides extremely simple conditions which do not involve matrix inversion or multiplication. Application of theorems 1-3 is straightforward for systems with symmetric stiffness matrices but is much more involved for systems with general stiffness matrices. One has to find a positive-definite symmetric matrix S which can symmetrize the stiffness matrix. This may not be easy. We proceed to avoid the necessity of finding the S matrix by proving a lemma which is later used to derive stability theorems 4 and 5 for systems with general stiffness matrices. Finally, two instability theorems and their three corollaries are derived.

Unlike many stability theorems which involve determination of the characteristic equations and their roots, or transformation to a system of first-order equations (thereby often losing sight of the physical parameters of the system), the theorems presented here yield conditions directly on the physical parameters of the system. This should prove very helpful to the analysts and designers dealing with multidimensional systems with time-varying and constant-parameter systems.

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